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Non-dispersive wavepacket solutions of the Schrödinger equation

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Abstract
The free Schrödinger equation has localized constant velocity solutions $\psi_{\mathbf{v}}$ of the form $\psi = f(\mathbf{r} - \mathbf{v}t)\,\mathrm{e}^{\mathrm{i}\lambda t}$ with λ being a constant. These solutions are eigenvectors of a momentum operator $\tilde{\mathbf{p}}$ such that $\tilde{\mathbf{p}}\psi_{\mathbf{v}} = m\mathbf{v}\psi_{\mathbf{v}}$. The wavepacket, while not normalizable, is both localized and in a definite momentum state. The $\psi_{\mathbf{v}}$ are orthogonal in the inner product space $\langle\phi|r^2|\psi\rangle$, and the $\tilde{\mathbf{p}}$ operator is symmetric therein. We discuss whether these $\psi_{\mathbf{v}}$ can act as basis states rather than the usual plane waves.

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1. Introduction

There has long been interest in localized solutions of Schrödinger's equation

$$(\mathrm{i}\hbar\partial_t + \hbar^2\nabla^2/2m)\psi = 0 \tag{1}$$

for modelling particles. Most attention has been given to the normalizable (finite energy) localized solutions, of which the best known are those with Gaussian envelopes, but these spread out over time. There are also non-dispersive localized solutions which have been given less attention: constant velocity wavepackets were discussed by Besieris *et al* [1] and Barut [2]. A constant acceleration wavepacket [3], sometimes called the Airy packet, is well known. These non-dispersive solutions have the drawback that they are not normalizable.

Here we introduce non-dispersive wavepacket solutions $\psi_{\mathbf{v}}$ of (1) with constant velocity $\mathbf{v}$ and amplitude inversely proportional to the distance from the wavepacket centre. The main result of this paper is to show that these $\psi_{\mathbf{v}}$ are eigenfunctions of a momentum operator $\tilde{\mathbf{p}}$, implying that a particle can be in a definite state of momentum while still being localized, in contrast to the plane wave eigenfunctions of the usual momentum operator $(-\mathrm{i}\hbar\boldsymbol{\nabla})$. The wavefunctions $\psi_{\mathbf{v}}$ are similar to those introduced in [1] (see (2.7) therein):

$$\psi_{\mathbf{v}} = \frac{\sin(mc|\mathbf{r}-\mathbf{v}t|/\hbar)}{|\mathbf{r}-\mathbf{v}t|}\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot(\mathbf{r}-\mathbf{v}t)/\hbar}\,\mathrm{e}^{-\mathrm{i}m(c^2-v^2)t/2\hbar} \tag{2}$$

with $v \equiv |\mathbf{v}|$ and c is an arbitrary parameter having dimension velocity, which can take any finite value. The extent of the wavefunction (2) about its centre ($\mathbf{r} = \mathbf{v}t$) is of order $\hbar/mc$, and as the Compton wavelength of a particle is $\lambda = 2\pi\hbar/mc$ where c is the velocity of light, we are minded to put c equals light velocity as suggested by the notation. (Any notion of particle size is problematic in quantum theory, but the Compton wavelength λ can be thought of as the minimum extent of a particle: to confine it within a distance smaller than λ results in additional particle, anti-particle pair production.) That $\psi_{\mathbf{v}} = f(\mathbf{r} - \mathbf{v}t)\,\mathrm{e}^{-\mathrm{i}m(c^2-v^2)t/2\hbar}$ is a clear statement of its non-dispersive property, the exponential being merely a time phase factor over all space.

To verify that (2) is a solution of (1), note the identities (we from now on put $\hbar = 1$)

$$\left(\partial_t + \mathbf{v}\cdot\nabla + \frac{\mathrm{i}m(c^2 - v^2)}{2}\right)\psi_{\mathbf{v}} = 0 \tag{3}$$

$$\begin{aligned}(\nabla - \mathrm{i}m\mathbf{v})\psi_{\mathbf{v}} &= \mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot(\mathbf{r}-\mathbf{v}t)}\,\mathrm{e}^{-\mathrm{i}m(c^2-v^2)t/2}\nabla\left[\frac{\sin(mc|\mathbf{r} - \mathbf{v}t|)}{|\mathbf{r} - \mathbf{v}t|}\right]\\ (\nabla - \mathrm{i}m\mathbf{v})^2\psi_{\mathbf{v}} &= \mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot(\mathbf{r}-\mathbf{v}t)}\,\mathrm{e}^{-\mathrm{i}m(c^2-v^2)t/2}\nabla^2\left[\frac{\sin(mc|\mathbf{r} - \mathbf{v}t|)}{|\mathbf{r} - \mathbf{v}t|}\right] = -m^2c^2\psi_{\mathbf{v}},\end{aligned} \tag{4}$$

the identity (3) follows from $\psi_{\mathbf{v}} = f(\mathbf{r} - \mathbf{v}t)\,\mathrm{e}^{-\mathrm{i}m(c^2-v^2)t/2}$. Expanding out (4) and substituting into (3) we obtain

$$\begin{aligned}&(\nabla^2 - 2\mathrm{i}m\mathbf{v}\cdot\nabla - m^2v^2)\psi_{\mathbf{v}} = -m^2c^2\psi_{\mathbf{v}}\\ &\left(\nabla^2 + 2\mathrm{i}m\left(\partial_t + \frac{\mathrm{i}m(c^2 - v^2)}{2}\right)\right)\psi_{\mathbf{v}} = -m^2(c^2 - v^2)\psi_{\mathbf{v}}\\ &(\nabla^2 + 2\mathrm{i}m\partial_t)\psi_{\mathbf{v}} = 0\end{aligned}$$

which is (1). A more direct method of verifying that $\psi_{\mathbf{v}}$ is a solution of (1) is to start with the stationary ($\mathbf{v} = 0$) wavepacket

$$\psi_{\mathbf{0}} = \frac{\sin(mcr)}{r}\,\mathrm{e}^{-\mathrm{i}mc^2t/2} \tag{5}$$

which is the spherical wave solution to Schrödinger's equation, it can be thought of as plane waves from all directions converging on and diverging from the origin. Then $\psi_{\mathbf{v}}$ may be obtained directly from $\psi_{\mathbf{0}}$ by a Galilei transformation $\mathcal{G}_{\mathbf{v}}$ which is defined [4]:

$$\mathcal{G}_{\mathbf{v}}\psi(\mathbf{r}, t) \equiv \psi(\mathbf{r} - \mathbf{v}t, t)\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}-\mathrm{i}mv^2t/2}. \tag{6}$$

Given a solution ψ of (1) then the boosted wavefunction $\psi' = \mathcal{G}_{\mathbf{v}}\psi$ is also a solution [4], and it is easily checked that

$$\mathcal{G}_{\mathbf{v}}\psi_{\mathbf{0}} = \psi_{\mathbf{v}}.$$

As already mentioned the $\psi_{\mathbf{v}}$ are not normalizable: consider the inner product of two wavepackets $\psi_{\mathbf{v}}, \psi_{\mathbf{v}'}$ momentarily coinciding at time $t = 0$

$$\begin{aligned}\langle\psi_{\mathbf{v}'}|\psi_{\mathbf{v}}\rangle &= \int(\sin(mcr)\,\mathrm{e}^{-\mathrm{i}m\mathbf{v}'\cdot\mathbf{r}}/r)(\sin(mcr)\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}}/r)\,\mathrm{d}^3\mathbf{r}\\ &= \pi\iint(1 - \cos(2mcr))\,\mathrm{e}^{\mathrm{i}m|\mathbf{v}-\mathbf{v}'|r\cos\theta}\mathrm{d}(-\cos\theta)\,\mathrm{d}r\\ &= \pi\int(1 - \cos(2mcr))\left[2\frac{\sin(m|\mathbf{v} - \mathbf{v}'|r)}{m|\mathbf{v} - \mathbf{v}'|r}\right]\mathrm{d}r\end{aligned}$$

$$= \frac{\pi}{m|\mathbf{v}-\mathbf{v}'|}\int\left[2\frac{\sin(m|\mathbf{v}-\mathbf{v}'|r)}{r} - \frac{\sin([2mc+m|\mathbf{v}-\mathbf{v}'|]r)}{r} + \frac{\sin([2mc-m|\mathbf{v}-\mathbf{v}'|]r)}{r}\right]\mathrm{d}r$$

$$= \frac{\pi}{m|\mathbf{v}-\mathbf{v}'|}\frac{\pi}{2}\left[2\,\mathrm{sgn}(m|\mathbf{v}-\mathbf{v}'|) - \mathrm{sgn}([2mc+m|\mathbf{v}-\mathbf{v}'|]) + \mathrm{sgn}([2mc-m|\mathbf{v}-\mathbf{v}'|])\right]$$

$$= \frac{\pi^2}{m|\mathbf{v}-\mathbf{v}'|} = \frac{\pi^2}{|\mathbf{p}-\mathbf{p}'|} \tag{7}$$

where in the final line of working we have put $\mathrm{sgn}([2mc - m|\mathbf{v}-\mathbf{v}'|]) = 1$ which holds for $|\mathbf{v}|, |\mathbf{v}'| < c$, bearing in mind that the Schrödinger equation is only valid for non-relativistic velocities. The $\langle\psi_{\mathbf{v}'}|\psi_{\mathbf{v}}\rangle$ is constant in time, so that two wavepackets which momentarily coincide have inner product which is a finite number (for $\mathbf{v} \neq \mathbf{v}'$) however far apart the wavepackets are separated (either before or after the wavepackets have coincided). This linkage between two coinciding wavepackets or particles is a well-known quantum feature. When $\mathbf{v} = \mathbf{v}'$ then $\langle\psi_{\mathbf{v}}|\psi_{\mathbf{v}}\rangle$ is infinite, though less singular than for the plane wave case.

The only free parameter in $\psi_{\mathbf{v}}$ is $\mathbf{v}$ itself which makes us curious as to whether $\psi_{\mathbf{v}}$ is an eigenfunction. We find a momentum operator $\tilde{\mathbf{p}}$ such that $\tilde{\mathbf{p}}\psi_{\mathbf{v}} = m\mathbf{v}\psi_{\mathbf{v}}$, and go on to consider whether we can regard the $\psi_{\mathbf{v}}$ as basis states of the Schrödinger equation rather than the usual plane wave solutions which are spread out over all space.

2. The momentum operator $\tilde{\mathbf{p}}$

In this section we put $t = 0$ so that the origin is at the wavepacket centre, and

$$\psi_{\mathbf{v}(t=0)} = \frac{\sin(mcr)}{r}\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}}. \tag{8}$$

Then

$$\left(\frac{1}{r}\nabla r\right)\left(\frac{\sin(mcr)}{r}\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}}\right) = \mathrm{i}\mathbf{p}\frac{\sin(mcr)}{r}\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}} + mc\hat{\mathbf{r}}\frac{\cos(mcr)}{r}\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}} \tag{9}$$

where $\hat{\mathbf{r}} \equiv \mathbf{r}/r$, which cannot immediately be resolved into an eigenvalue equation due to the cosine function on the rhs instead of a sine function. As is well known the Hilbert transform operator $\mathcal{H}$ defined by

$$g(x) = \mathcal{H}f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t-x}\,\mathrm{d}t \tag{10}$$

exchanges cosine and sine functions, as

$$\mathcal{H}\mathrm{e}^{\mathrm{i}\lambda x} = \mathrm{i}\,\mathrm{sgn}(\lambda)\,\mathrm{e}^{\mathrm{i}\lambda x}.$$

We will adapt the Hilbert transform operator to 3D space by integrating along the entire axis through the origin and $\mathbf{r}$, but we cannot simply follow (10) and write $g(r,\theta,\phi) = \mathcal{H}f(r,\theta,\phi)$ because $r \equiv |\mathbf{r}|$ is non-negative. We will use the parity operator

$$\mathcal{P}f(x) \equiv f(-x)$$

to supply the negative part in the integration range of (10).

2.1. The operators $\mathcal{H}_\pm$

First we note that (10) can be written as

$$\mathcal{H}f(x) = \frac{1}{\pi}\int_0^\infty \left(\frac{f(t)}{t-x} - \frac{f(-t)}{t+x}\right)\mathrm{d}t$$
$$= \frac{1}{\pi}x\int_0^\infty \frac{f(t)+f(-t)}{t^2-x^2}\,\mathrm{d}t + \frac{1}{\pi}\int_0^\infty t\frac{f(t)-f(-t)}{t^2-x^2}\,\mathrm{d}t$$
$$\equiv \mathcal{H}_e f_e(x) + \mathcal{H}_o f_o(x) \tag{11}$$

where $f_e(x), f_o(x)$ are the even, odd parts of $f(x)$ which we can write in terms of the projection operators $\mathcal{P}_\pm$:

$$f_e(x) \equiv \mathcal{P}_+ f(x) \equiv \frac{1+\mathcal{P}}{2}f(x), \qquad f_o(x) \equiv \mathcal{P}_- f(x) \equiv \frac{1-\mathcal{P}}{2}f(x), \tag{12}$$

and $\mathcal{H}_e, \mathcal{H}_o$ (known as the Hilbert transforms of even/odd functions) are defined

$$\mathcal{H}_e f(x) = \frac{2x}{\pi}\int_0^\infty \frac{f(t)}{t^2-x^2}\,\mathrm{d}t, \qquad \mathcal{H}_o f(x) = \frac{2}{\pi}\int_0^\infty \frac{tf(t)}{t^2-x^2}\,\mathrm{d}t. \tag{13}$$

Returning to (11) we can write $\mathcal{H}$ in terms of $\mathcal{H}_e, \mathcal{H}_o$ and $\mathcal{P}_\pm$ as follows:

$$\mathcal{H} = \mathcal{H}_e\mathcal{P}_+ + \mathcal{H}_o\mathcal{P}_-. \tag{14}$$

By a simple extension of (11)–(14) we can define the following operators $\mathcal{H}_\pm$ valid in 3D space:

$$\begin{aligned}\mathcal{H}_+ f(r,\theta,\phi) &\equiv (\mathcal{H}_e\mathcal{P}_+ + \mathcal{H}_o\mathcal{P}_-)f(r,\theta,\phi)\\ \mathcal{H}_- f(r,\theta,\phi) &\equiv (\mathcal{H}_e\mathcal{P}_- + \mathcal{H}_o\mathcal{P}_+)f(r,\theta,\phi)\end{aligned} \tag{15}$$

where $\mathcal{P}f(r,\theta,\phi) = f(r,\pi-\theta,\phi+\pi)$,

$$\mathcal{H}_e f(r,\theta,\phi) \equiv \frac{2r}{\pi}\int_0^\infty \frac{f(t,\theta,\phi)}{t^2-r^2}\,\mathrm{d}t, \qquad \mathcal{H}_o f(r,\theta,\phi) \equiv \frac{2}{\pi}\int_0^\infty \frac{tf(t,\theta,\phi)}{t^2-r^2}\,\mathrm{d}t; \tag{16}$$

equivalently $\mathcal{P}f(\mathbf{r}) = f(-\mathbf{r})$,

$$\mathcal{H}_e f(\mathbf{r}) = \frac{2}{\pi}\int_0^\infty \frac{f(\lambda\mathbf{r})}{\lambda^2-1}\,\mathrm{d}\lambda, \qquad \mathcal{H}_o f(\mathbf{r}) = \frac{2}{\pi}\int_0^\infty \frac{\lambda f(\lambda\mathbf{r})}{\lambda^2-1}\,\mathrm{d}\lambda.$$

Note that the parity operator commutes with the $\mathcal{H}_e, \mathcal{H}_o$ operators of (16).

We now verify that $\{\cos(mcr)\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}}\} = \mathcal{H}_-\{\sin(mcr)\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}}\}$ which result will enable us to construct the momentum operator. First we note the identities

$$\mathcal{H}_e\cos(\lambda r) = -\mathrm{sgn}(\lambda)\sin(\lambda r), \qquad \mathcal{H}_o\sin(\lambda r) = \mathrm{sgn}(\lambda)\cos(\lambda r) \tag{17}$$

then

$$\begin{aligned}&\mathcal{H}_-\{\sin(mcr)\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}}\}\\ &\quad\equiv (\mathcal{H}_e\mathcal{P}_- + \mathcal{H}_o\mathcal{P}_+)\{\sin(mcr)\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}}\}\\ &\quad= \mathcal{H}_e\{\mathrm{i}\sin(mcr)\sin(m\mathbf{v}\cdot\mathbf{r})\} + \mathcal{H}_o\{\sin(mcr)\cos(m\mathbf{v}\cdot\mathbf{r})\}\\ &\quad= \tfrac{1}{2}[\mathrm{i}\mathcal{H}_e\{\cos(\{mc - m\mathbf{v}\cdot\hat{\mathbf{r}}\}r) - \cos(\{mc + m\mathbf{v}\cdot\hat{\mathbf{r}}\}r)\}\\ &\qquad+ \mathcal{H}_o\{\sin(\{mc + m\mathbf{v}\cdot\hat{\mathbf{r}}\}r) + \sin(\{mc - m\mathbf{v}\cdot\hat{\mathbf{r}}\}r)\}]\end{aligned}$$

$$= \tfrac{1}{2}[-\mathrm{i}\sin(\{mc - m\mathbf{v}\cdot\hat{\mathbf{r}}\}r) + \mathrm{i}\sin(\{mc + m\mathbf{v}\cdot\hat{\mathbf{r}}\}r)\}$$
$$+ \cos(\{mc + m\mathbf{v}\cdot\hat{\mathbf{r}}\}r) + \cos(\{mc - m\mathbf{v}\cdot\hat{\mathbf{r}}\}r)]$$
$$= \tfrac{1}{2}[\mathrm{e}^{\mathrm{i}(mcr + m\mathbf{v}\cdot\mathbf{r})} + \mathrm{e}^{-\mathrm{i}(mcr - m\mathbf{v}\cdot\mathbf{r})}]$$
$$= \cos(mcr)\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}} \tag{18}$$

as stated. In the working above we have used the fact that the factors $\{mc \pm m\mathbf{v}\cdot\hat{\mathbf{r}}\}$ multiplying r are positive definite for $|\mathbf{v}| < c$. We now substitute (18) into (8) obtaining

$$\left(\frac{1}{r}\nabla r\right)\left(\frac{\sin(mcr)}{r}\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}}\right) = \mathrm{i}\mathbf{p}\left(\frac{\sin(mcr)}{r}\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}}\right) + mc\frac{\hat{\mathbf{r}}}{r}\mathcal{H}_{-}r\left(\frac{\sin(mcr)}{r}\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}}\right)$$
$$-\mathrm{i}\frac{1}{r}(\nabla - mc\hat{\mathbf{r}}\mathcal{H}_{-})r\psi_{\mathbf{v}}(t=0) = \mathbf{p}\psi_{\mathbf{v}}(t=0)$$

so that the momentum operator is

$$\tilde{\mathbf{p}} = -\mathrm{i}\frac{1}{r}(\nabla - mc\hat{\mathbf{r}}\mathcal{H}_{-})r = -\mathrm{i}\nabla - \mathrm{i}\frac{\hat{\mathbf{r}}}{r} - \mathrm{i}\frac{mc\hat{\mathbf{r}}}{r}\mathcal{H}_{-}r. \tag{19}$$

3. That $\tilde{\mathbf{p}}$ is symmetric in $\langle\phi|r^2|\psi\rangle$

Inspection of the time ($t = 0$) wavefunctions (8) suggests that $\psi_{\mathbf{v}}$, $\psi_{\mathbf{v}'}$ may be orthogonal in the r^2 inner product space

$$\langle\phi|r^2|\psi\rangle = \langle r\phi|r\psi\rangle \equiv \int (r^2\phi^*\psi)\,\mathrm{d}^3\mathbf{r}. \tag{20}$$

We first prove this orthogonality, and then go on to show that the $\tilde{\mathbf{p}}$ operator of (19) is symmetric with respect to $\langle\phi|r^2|\psi\rangle$.

We recall the well-known identity

$$\int \mathrm{e}^{\mathrm{i}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}}\,\mathrm{d}^3\mathbf{r} = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}') \tag{21}$$

where the rhs is the 3D delta function. Then the inner product $\langle\psi_{\mathbf{v}'}|r^2|\psi_{\mathbf{v}}\rangle$ of two wavepackets momentarily coinciding is

$$\langle r\psi_{\mathbf{v}'}|r\psi_{\mathbf{v}}\rangle = \int \sin^2(mcr)\,\mathrm{e}^{\mathrm{i}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}}\,\mathrm{d}^3\mathbf{r}$$
$$= \frac{1}{2}\int (\mathrm{e}^{\mathrm{i}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}} - \cos(2mcr)\,\mathrm{e}^{\mathrm{i}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}})\,\mathrm{d}^3\mathbf{r}$$
$$= \pi\iint (\mathrm{e}^{\mathrm{i}|\mathbf{p}-\mathbf{p}'|r\cos\theta} - \cos(2mcr)\,\mathrm{e}^{\mathrm{i}|\mathbf{p}-\mathbf{p}'|r\cos\theta})\mathrm{d}(-\cos\theta)r^2\,\mathrm{d}r$$
$$= \frac{2\pi}{|\mathbf{p}-\mathbf{p}'|}\int (r\sin(|\mathbf{p}-\mathbf{p}'|r) - r\cos(2mcr)\sin(|\mathbf{p}-\mathbf{p}'|r))\,\mathrm{d}r$$
$$= \frac{\pi}{|\mathbf{p}-\mathbf{p}'|}\int (2r\sin(|\mathbf{p}-\mathbf{p}'|r) - r\sin[(|\mathbf{p}-\mathbf{p}'| + 2mc)r]$$
$$- r\sin[(|\mathbf{p}-\mathbf{p}'| - 2mc)r])\,\mathrm{d}r. \tag{22}$$

We know from (21) that the value of the first integral in (22) is zero for $|\mathbf{p}-\mathbf{p}'| \neq 0$, which implies the improper integral identity

$$\int_0^\infty r\sin(\lambda r)\,\mathrm{d}r = 0 \qquad \text{for} \quad \lambda \neq 0.$$

And as $(|\mathbf{p}-\mathbf{p}'| \pm 2mc) \neq 0$ for $|\mathbf{v}|, |\mathbf{v}'| < c$, the second and third integrals of (22) are zero, and

$$\langle \psi_{\mathbf{v}'} | r^2 | \psi_{\mathbf{v}} \rangle = \langle r\psi_{\mathbf{v}'} | r\psi_{\mathbf{v}} \rangle = 4\pi^3 \delta(\mathbf{p}-\mathbf{p}'). \tag{23}$$

The result (23) may be used to project out the $\psi_{\mathbf{v}'}$ component in a superposition of $\psi_{\mathbf{v}}$.

As the $\psi_{\mathbf{v}}$ are eigenfunctions of the $\tilde{\mathbf{p}}$ operator of (19), the result (23) implies that $\tilde{\mathbf{p}}$ is symmetric with respect to the r^2 inner product space (20), which we will now verify. The first component of $\tilde{\mathbf{p}}$ is $\left(-\mathrm{i}\frac{1}{r}\nabla r\right) = (-\mathrm{i}\nabla - \mathrm{i}\hat{\mathbf{r}}/r)$, and

$$\left\langle \phi | r^2 | -\mathrm{i}\frac{1}{r}\nabla r\psi \right\rangle = \langle r\phi | -\mathrm{i}\nabla r\psi \rangle = \langle -\mathrm{i}\nabla r\phi | r\psi \rangle = \left\langle -\mathrm{i}\frac{1}{r}\nabla r\phi | r^2 | \psi \right\rangle. \tag{24}$$

The remaining component of $\tilde{\mathbf{p}}$ is the operator $\left(\mathrm{i}mc\frac{1}{r}\hat{\mathbf{r}}\mathcal{H}_- r\right)$, and

$$\left\langle \phi | r^2 | \mathrm{i}\frac{1}{r}\hat{\mathbf{r}}\mathcal{H}_- r\psi \right\rangle = \langle r\phi | \mathrm{i}\hat{\mathbf{r}}\mathcal{H}_- r\psi \rangle. \tag{25}$$

With the substitutions

$$\Psi \equiv r\psi, \qquad \Phi \equiv r\phi \tag{26}$$

then (25) is $\langle \Phi | \mathrm{i}\hat{\mathbf{r}}\mathcal{H}_- \Psi \rangle$. In the working below we write $\Psi = \Psi(r)$, the angular variables θ, ϕ being understood, and $\mathrm{d}\Omega \equiv \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$:

$$\begin{aligned}
\langle \Phi | \mathrm{i}\hat{\mathbf{r}}\mathcal{H}_- \Psi \rangle &\equiv \iint \Phi^*(r) \left[\mathrm{i}\hat{\mathbf{r}} \frac{1}{\pi} \int_0^\infty \left(\frac{1}{t-r}\Psi(t) + \frac{1}{t+r}\mathcal{P}\Psi(t) \right) \mathrm{d}t \right] r^2 \, \mathrm{d}r \, \mathrm{d}\Omega \\
&= \frac{1}{\pi} \int_\Omega \int_{r=0}^\infty \int_{t=0}^\infty \mathrm{i}\hat{\mathbf{r}}\Phi^*(r) \left(\frac{t^2 - (t^2 - r^2)}{t-r}\Psi(t) \right. \\
&\quad \left. + \frac{t^2 - (t^2 - r^2)}{t+r}\mathcal{P}\Psi(t) \right) \mathrm{d}t \, \mathrm{d}r \, \mathrm{d}\Omega \\
&= \int_\Omega \int_{t=0}^\infty \left[\frac{1}{\pi} \int_{r=0}^\infty \left(-\frac{1}{r-t}\mathrm{i}\hat{\mathbf{r}}\Phi^*(r) + \frac{1}{r+t}\mathcal{P}\mathrm{i}\hat{\mathbf{r}}\Phi^*(r) \right) \mathrm{d}r \right] t^2 \Psi(t) \, \mathrm{d}t \, \mathrm{d}\Omega \\
&\quad - \frac{1}{\pi} \int_\Omega \int_{r=0}^\infty \int_{t=0}^\infty \mathrm{i}\hat{\mathbf{r}}[(t+r)\Phi^*(r)\Psi(t) + (t-r)\Phi^*(r)\mathcal{P}\Psi(t)] \, \mathrm{d}t \, \mathrm{d}r \, \mathrm{d}\Omega \\
&= \int_\Omega \int_0^\infty \left[\mathrm{i}\hat{\mathbf{r}} \frac{1}{\pi} \int_{r=0}^\infty \left(-\frac{1}{r-t}\Phi^*(r) - \frac{1}{r+t}\mathcal{P}\Phi^*(r) \right) \mathrm{d}r \right] t^2 \Psi(t) \, \mathrm{d}t \, \mathrm{d}\Omega \\
&\quad - \frac{1}{\pi} \int_\Omega \mathrm{i}\hat{\mathbf{r}} \left\{ \left[\int_0^\infty \Phi^*(r) \mathrm{d}r \right] \left[\int_0^\infty (t\Psi(t) + t\mathcal{P}\Psi(t)) \, \mathrm{d}t \right] \right\} \mathrm{d}\Omega \\
&\quad - \frac{1}{\pi} \int_\Omega \mathrm{i}\hat{\mathbf{r}} \left\{ \left[\int_0^\infty r\Phi^*(r) \, \mathrm{d}r \right] \left[\int_0^\infty (\Psi(t) - \mathcal{P}\Psi(t)) \, \mathrm{d}t \right] \right\} \mathrm{d}\Omega \\
&= \langle \mathrm{i}\hat{\mathbf{r}}\mathcal{H}_- \Phi | \Psi \rangle - \frac{1}{\pi} \int_\Omega \mathrm{i}\hat{\mathbf{r}} \left\{ \left[\int_0^\infty (\Phi^*(r) - \mathcal{P}\Phi^*(r)) \, \mathrm{d}r \right] \left[\int_0^\infty t\Psi(t) \mathrm{d}t \right] \right\} \mathrm{d}\Omega \\
&\quad - \frac{1}{\pi} \int_\Omega \mathrm{i}\hat{\mathbf{r}} \left\{ \left[\int_0^\infty r\Phi^*(r) \, \mathrm{d}r \right] \left[\int_0^\infty (\Psi(t) - \mathcal{P}\Psi(t)) \, \mathrm{d}t \right] \right\} \mathrm{d}\Omega.
\end{aligned} \tag{27}$$

In the working above we have used the self-adjoint property of $\mathcal{P}$, and that $\mathcal{P}$ anticommutes with $\hat{\mathbf{r}}$. We see that if Ψ, Φ satisfy the boundary condition

$$\mathcal{I}\Psi \equiv \int_0^\infty (\Psi(r) - \mathcal{P}\Psi(r))\,\mathrm{d}r = 0 \tag{28}$$

then

$$\langle \Phi | \mathrm{i}\hat{\mathbf{r}}\mathcal{H}_-\Psi\rangle = \langle \mathrm{i}\hat{\mathbf{r}}\mathcal{H}_-\Phi|\Psi\rangle, \tag{29}$$

and $(\mathrm{i}\hat{\mathbf{r}}\mathcal{H}_-)^\dagger = (\mathrm{i}\hat{\mathbf{r}}\mathcal{H}_-)$, so subject to (28) the $\tilde{\mathbf{p}}$ operator is symmetric with respect to the inner product space $\langle \phi | r^2 | \psi\rangle$. The boundary condition (28) is satisfied by the $\Psi_{\mathbf{v}}$ as

$$\mathcal{I}\Psi_{\mathbf{v}} \equiv \mathcal{I}r\psi_{\mathbf{v}} = \mathrm{i}\int_0^\infty \sin(mcr)\sin(m(\mathbf{v}\cdot\hat{\mathbf{r}})r)\,\mathrm{d}r = \frac{\mathrm{i}\pi}{m}\delta(c - \mathbf{v}\cdot\hat{\mathbf{r}})$$

which is zero for $v < c$. On this and other occasions in our working we have found the necessity of the light speed limit for v.

4. The Fourier transform of $\psi_{\mathbf{v}}$

We can express $\psi_{\mathbf{v}}$ as a superposition of plane waves: the Fourier transform $\phi_{\mathbf{v}}(\mathbf{k})$ of $\psi_{\mathbf{v}}$ may be found via the following steps. With

$$\mathcal{F}[\phi(\mathbf{k})] \equiv \frac{1}{(2\pi)^{3/2}}\int \phi(\mathbf{k})\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\mathrm{d}\mathbf{k} = \psi(\mathbf{r})$$

and the 3D Fourier transform of spherically symmetric functions

$$\mathcal{F}[\phi(k)] = \sqrt{\frac{2}{\pi}}\frac{1}{r}\int \phi(k)\sin(kr)k\,\mathrm{d}k = \psi(r),$$

then

$$\mathcal{F}\left[\sqrt{\frac{\pi}{2}}\frac{\delta(k - mc)}{mc}\right] = \frac{\sin(mcr)}{r}$$

$$\mathcal{F}\left[\sqrt{\frac{\pi}{2}}\frac{\delta(|\mathbf{k} - m\mathbf{v}| - mc)}{mc}\right] = \frac{\sin(mcr)}{r}\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}}$$

$$\mathcal{F}\left[\sqrt{\frac{\pi}{2}}\,\mathrm{e}^{-\mathrm{i}t\mathbf{v}\cdot\mathbf{k}}\frac{\delta(|\mathbf{k} - m\mathbf{v}| - mc)}{mc}\right] = \frac{\sin(mc|\mathbf{r} - \mathbf{v}t|)}{|\mathbf{r} - \mathbf{v}t|}\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot(\mathbf{r}-\mathbf{v}t)}$$

$$\begin{aligned}\mathcal{F}[\phi_{\mathbf{v}}] &\equiv \mathcal{F}\left[\sqrt{\frac{\pi}{2}}\,\mathrm{e}^{-\mathrm{i}t\mathbf{v}\cdot\mathbf{k}-\mathrm{i}m(c^2-v^2)t/2}\frac{\delta(|\mathbf{k} - m\mathbf{v}| - mc)}{mc}\right]\\ &= \frac{\sin(mc|\mathbf{r} - \mathbf{v}t|)}{|\mathbf{r} - \mathbf{v}t|}\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot(\mathbf{r}-\mathbf{v}t)}\,\mathrm{e}^{-\mathrm{i}m(c^2-v^2)t/2} \equiv \psi_{\mathbf{v}},\end{aligned} \tag{30}$$

so that the momentum space wavefunction $\phi_{\mathbf{v}}(\mathbf{k})$ is

$$\begin{aligned}\phi_{\mathbf{v}} &= \sqrt{\frac{\pi}{2}}\,\mathrm{e}^{-\mathrm{i}t\mathbf{v}\cdot\mathbf{k}-\mathrm{i}m(c^2-v^2)t/2}\frac{\delta(|\mathbf{k} - m\mathbf{v}| - mc)}{mc}\\ &= \sqrt{\frac{\pi}{2}}\,\mathrm{e}^{-\mathrm{i}k^2t/2m}\frac{\delta(|\mathbf{k} - m\mathbf{v}| - mc)}{mc}.\end{aligned} \tag{31}$$

For the last step of working, note that the delta function $\delta(|\mathbf{k} - m\mathbf{v}| - mc)$ is non-zero on the surface $k^2 - 2m(\mathbf{v}\cdot\mathbf{k}) + m^2v^2 - m^2c^2 = 0$. This surface is a sphere of radius mc whose centre is displaced from the origin by $m\mathbf{v}$. Assuming that $|\mathbf{v}| < c$ the origin is inside this displaced sphere; so like the stationary wavefunction $\psi_{\mathbf{v}}$ is a superposition of plane waves from all directions.

While we have been dealing with the usual three-dimensional case, the $\phi_{\mathbf{v}}(\mathbf{k})$ of (31) is applicable to n dimensions with $\mathbf{k} \equiv (k_1, \ldots, k_n)$, $\mathbf{v} \equiv (v_1, \ldots, v_n)$, then the corresponding $\psi_{\mathbf{v}(n)}$ solution to Schrödinger's equation in n dimensions can be shown to be

$$\psi_{\mathbf{v}(n)} = \left(\frac{\pi}{2mc|\mathbf{r} - \mathbf{v}t|}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(mc|\mathbf{r} - \mathbf{v}t|)\, \mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot(\mathbf{r}-\mathbf{v}t)}\, \mathrm{e}^{-\mathrm{i}m(c^2-v^2)t/2} \tag{32}$$

with $\mathbf{r} \equiv (r_1, \ldots, r_n)$, and J is the Bessel function. The one- and two-dimensional cases are

$$\psi_{v(1)} = 2\cos(mc(r - vt))\, \mathrm{e}^{\mathrm{i}mv(r-vt)}\, \mathrm{e}^{-\mathrm{i}m(c^2-v^2)t/2}$$
$$\psi_{\mathbf{v}(2)} = J_0(mc|\mathbf{r} - \mathbf{v}t|)\, \mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot(\mathbf{r}-\mathbf{v}t)}\, \mathrm{e}^{-\mathrm{i}m(c^2-v^2)t/2}.$$

5. Outlook

Our strategy in this paper has been to look for non-dispersive, constant velocity solutions $\psi_{\mathbf{v}}$ of the Schrödinger equation, then we find a momentum operator such that $\tilde{\mathbf{p}}\psi_{\mathbf{v}} = m\mathbf{v}\psi_{\mathbf{v}}$. It is a postulate of quantum mechanics that dynamical variables can be represented by operators whose eigenfunctions have definite values of that variable, and in our case the wavepacket $\psi_{\mathbf{v}}$ clearly does has momentum $m\mathbf{v}$, being in the form $f(\mathbf{r} - \mathbf{v}t)$ multiplied by a phase factor. In contrast the plane wave with momentum $\mathbf{p}$ has no such obvious velocity. Secondly the $\tilde{\mathbf{p}}$ operator allows a particle to both be localized and in a definite state of momentum, instead of being spread out over all space. Our approach implies that the Schrödinger equation is fundamental whereas the usual momentum operator $(-\mathrm{i}\nabla)$ is not. In this context we note that the Schrödinger equation anteceded the momentum operator, being derived from classical variational principles (for an interesting discussion of Schrödinger's derivation, see chapter 3 of [5] and references therein).

The orthogonality relation (23) suggests the $\psi_{\mathbf{v}}$ as alternative basis states rather than the usual plane waves, however the price to be paid is the greater complexity of the $\tilde{\mathbf{p}}$ operator, and it only has the eigenfunction property when the origin is at the wavepacket centre. Other non-dispersive solutions of (1) can be constructed, we introduce a particular class in the appendix, but we conjecture that the solutions (2) are unique in that they are form invariant under a Galileo boost transformation $\mathcal{G}_{\mathbf{u}}$

$$\mathcal{G}_{\mathbf{u}}\psi(\mathbf{r}, t) \equiv \psi(\mathbf{r} - \mathbf{u}t, t)\, \mathrm{e}^{\mathrm{i}m\mathbf{u}\cdot\mathbf{r}-imu^2t/2},$$

as by inspection

$$\mathcal{G}_{\mathbf{u}}\psi_{\mathbf{v}} = \psi_{\mathbf{v}+\mathbf{u}}.$$

We recall from section 1 that this identity can be used to derive the $\psi_{\mathbf{v}}$ from the stationary wavefunction $\psi_{\mathbf{0}}$.

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Appendix

A further class of solutions to Schrödinger's equation is

$$\chi_{\mathbf{v}} = \frac{\sin(mv^0|\mathbf{r} - \mathbf{v}t|)}{|\mathbf{r} - \mathbf{v}t|}\, \mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot(\mathbf{r}-\mathbf{v}t)}\, \mathrm{e}^{-\mathrm{i}mc^2t/2} \tag{A.1}$$

with $v^0 \equiv (c^2 + v^2)^{1/2}$. That the $\chi_{\mathbf{v}}$ is a solution of (1) may be verified similarly to the methods of section 1. Note that $mcv^0 = mc(c^2 + v^2)^{1/2} \approx mc^2 + \frac{1}{2}mv^2$ for $v \ll c$, which is the non-relativistic energy $\frac{1}{2}mv^2$ plus a 'rest energy' mc^2.

At time $t = 0$

$$\chi_{\mathbf{v}} = \frac{\sin(mv^0 r)}{r}\,\mathrm{e}^{\mathrm{i}m\mathbf{v}\cdot\mathbf{r}} \tag{A.2}$$

which is an eigenfunction of a momentum operator $\check{\mathbf{p}}$ with eigenvalue $\mathbf{p} = m\mathbf{v}$, which we will now demonstrate. First note the following identities, which can be verified by direct calculation:

$$\left[-(\partial_r r)\boldsymbol{\nabla} - \tfrac{1}{2}\mathbf{r}(m^2c^2 - \nabla^2)\right]\{\sin(p^0 r)\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}}\} = -\mathrm{i}(\partial_r r)\mathbf{p}\{\sin(p^0 r)\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}}\} \tag{A.3}$$

$$\begin{aligned}\left(\tfrac{1}{2}m^2c^2 r + a^0\right)\{\sin(p^0 r)\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}}\} &= -(\partial_r r)p^0\{\cos(p^0 r)\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}}\}\\ &= -(\partial_r r)p^0\mathcal{H}_-\{\sin(p^0 r)\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}}\}\end{aligned} \tag{A.4}$$

where the operator $(\partial_r r)$ is short for $(r\partial_r + \mathbf{1})$ and $p^0 \equiv mv^0$. We define the operators $\mathbf{a}$, and a^0 with $(a^0)^2 = \mathbf{a}^2$:

$$(a^0, \mathbf{a}) = \left(-\tfrac{1}{2}r\nabla^2, -(\partial_r r)\boldsymbol{\nabla} + \tfrac{1}{2}\mathbf{r}\nabla^2\right), \tag{A.5}$$

the operator $\mathbf{a}$ is related to the Runge–Lenz operator used for solving the Schrödinger equation with a Coulomb potential, its components commute with each other. Further properties of $(a^0, \mathbf{a})$ are listed in [7]. We define the dilation operator D and its inverse:

$$D \equiv -\mathrm{i}\partial_r r = -\mathrm{i}(r\partial_r + \mathbf{1}), \qquad D^{-1}f(r,\theta,\phi) = \frac{\mathrm{i}}{r}\int_0^r f(t,\theta,\phi)\,\mathrm{d}t. \tag{A.6}$$

We can now write (A.3), (A.4) as

$$\left(-\tfrac{1}{2}m^2c^2\mathbf{r} + \mathbf{a}\right)r\chi_{\mathbf{v}} = Dr\mathbf{p}\chi_{\mathbf{v}} \tag{A.7}$$

$$\left(\tfrac{1}{2}m^2c^2 r + a^0\right)r\chi_{\mathbf{v}} = -\mathrm{i}D\mathcal{H}_- r p^0\chi_{\mathbf{v}} \tag{A.8}$$

and multiplying from the left by $\frac{1}{r}D^{-1}$ we obtain

$$\check{\mathbf{p}}\chi_{\mathbf{v}} \equiv \tfrac{1}{r}D^{-1}\left(-\tfrac{1}{2}m^2c^2\mathbf{r} + \mathbf{a}\right)r\chi_{\mathbf{v}} = \mathbf{p}\chi_{\mathbf{v}}, \tag{A.9}$$

$$\check{p}^0\chi_{\mathbf{v}} \equiv -\mathrm{i}\tfrac{1}{r}\mathcal{H}_+D^{-1}\left(\tfrac{1}{2}m^2c^2 r + a^0\right)r\chi_{\mathbf{v}} = p^0\chi_{\mathbf{v}} \tag{A.10}$$

where in the last step we have used $\mathcal{H}_+\mathcal{H}_- = -\mathbf{1}$.

As $p^{0^2} = m^2 + \mathbf{p}^2$ these operators have relativistic character, also the Galileo transformation does not preserve the form of $\chi_{\mathbf{v}}$. For these reasons we think these operators are more suited to the relativistic case: elsewhere we will consider a modified $\chi_{\mathbf{v}}$ which satisfies a relativistic evolution equation.

References

[1] Besieres I M *et al* 1994 *Am. J. Phys.* **62** 519
[2] Barut A O 1990 *Found. Phys.* **20** 1233
[3] Berry M V and Balazs N L 1979 *Am. J. Phys.* **47** 264
[4] Ballentine L E 1990 *Quantum Mechanics* (Englewood Cliffs, NJ: Prentice-Hall) pp 78–80
[5] Bialynicki-Birula I *et al* 1990 *Theory of Quanta* (New York: Oxford University Press)
[6] Kaiser G 2008 private communication
[7] Mosley S N 1996 *J. Phys. A: Math. Gen.* **29** 6671